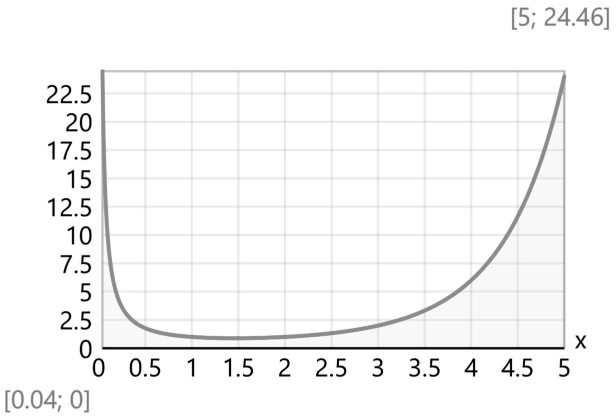
Evaluation of special math functions in Calcpad

This workheet defines some of the most common special functions in $\mathbb{R}$, by using the existing numerical methods in Calcpad only in stable and precise way (as possible)

Gamma and related functions

Euler-Mascheroni constant - $\gamma = 0.577$

Gamma function - $\Gamma(x) = \frac{1}{x} \cdot \int_0^1 (-\ln(t))^x \, dt$

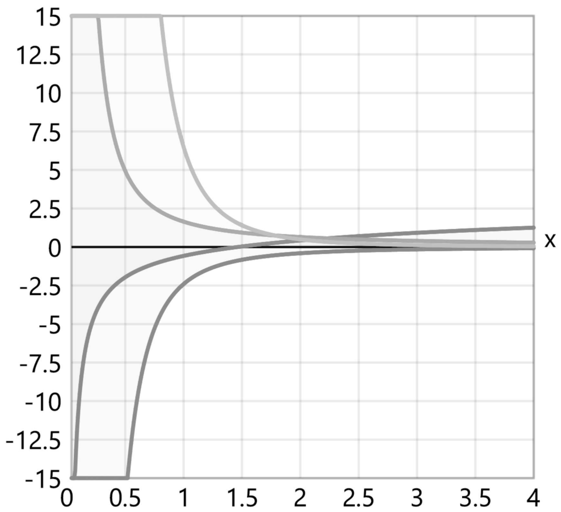


Checks: $\Gamma(0) = +\infty$, $\Gamma(4) = 6 = 3!$, $\Gamma(5) = 24 = 4!$

$x = 50$, $x! = 50! = 3.04\times10^{64}$, $\frac{\Gamma(x+1) - x!}{x!} = \frac{\Gamma(50+1) - 50!}{50!} = -3.84\times10^{-16}$

Digamma function - $\psi(x) = \int_0^1 \frac{1 - t^{x-1}}{1 - t} \, dt - \gamma$

Polygamma function - $\psi_{\mathrm{m}}(m; x) = (-1)^{m+1} \cdot \int_\varepsilon^1 \frac{(-\ln(t))^m \cdot t^{x-1}}{1 - t} \, dt$, where $\varepsilon = 10^{-300}$



Checks:

$\psi_{\mathrm{m}}(1; 1) - \dfrac{\pi^2}{6} = \psi_{\mathrm{m}}(1; 1) - \dfrac{3.14^2}{6} = 2.22\times10^{-16}$

$\psi_{2,1}$ = -2.4 -2 times Apéry constant

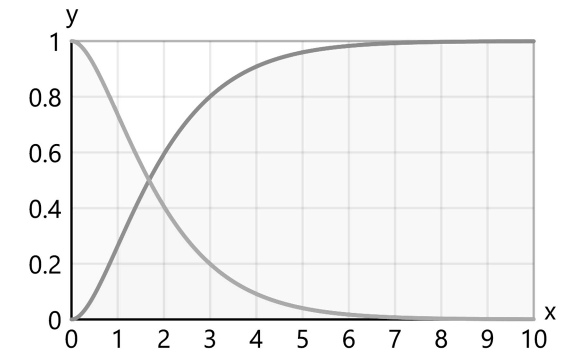
$\psi_{\mathrm{m}}(2; 1) - \psi_{2,1} = \psi_{\mathrm{m}}(2; 1) - (-2.4) = -8.44\times10^{-15}$

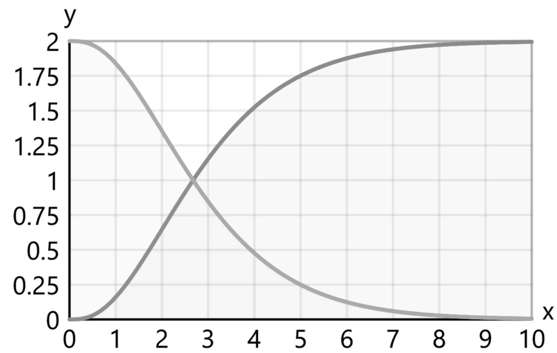
$\psi_{\mathrm{m}}(3; 1) - \dfrac{\pi^4}{15} = \psi_{\mathrm{m}}(3; 1) - \dfrac{3.14^4}{15} = 2.66\times10^{-15}$

Incomplete Gamma functions:

$\gamma(s; x) = \displaystyle\int_{e^{-x}}^{1} (-\mathbf{ln}(t))^{s-1}\,\mathrm{d}t$ or $\gamma(s; x) = \dfrac{1}{s}\cdot\left(\displaystyle\int_{e^{-x}}^{1} (-\mathbf{ln}(t))^{s}\,\mathrm{d}t + x^{s}\cdot e^{-x}\right)$

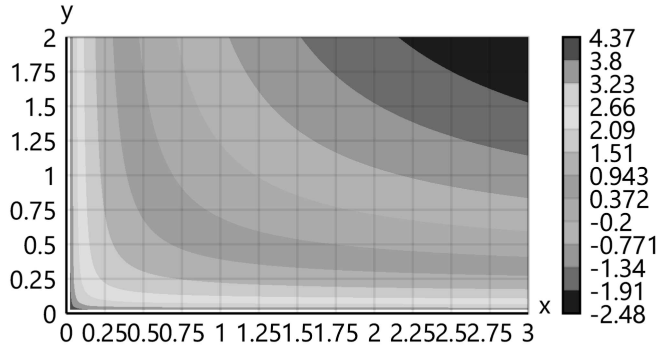
$\mathit{\Gamma}\_(s; x) = \displaystyle\int_{0}^{e^{-x}} (-\mathbf{ln}(t))^{s-1}\,\mathrm{d}t$ or $\mathit{\Gamma}\_(s; x) = \dfrac{1}{s}\cdot\left(\displaystyle\int_{0}^{e^{-x}} (-\mathbf{ln}(t))^{s}\,\mathrm{d}t - x^{s}\cdot e^{-x}\right)$





Beta function - $B(x; y) = \dfrac{\mathit{\Gamma}(x)\cdot\mathit{\Gamma}(y)}{\mathit{\Gamma}(x+y)}$

[3; 2]



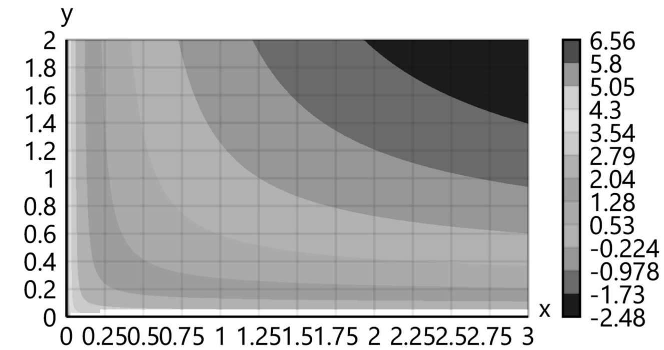
[0; 0]

Checks: $B(3; 2) \cdot 12 = 1$, $B(4; 3) \cdot 60 = 1$

Incomplete Beta function - $B(x; a; b) = \int_{\varepsilon}^{x} t^{a-1} \cdot (1 - t)^{b-1} \, \mathrm{d}t$

$Precision = 10^{-6}$

[3; 2]

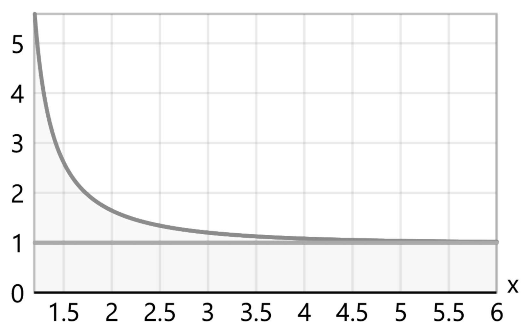


[0.0001; 0.0001]

Checks: $B(1; 1; 1) = 1$, $B(1; 2; 3) \cdot 12 = 1$, $\frac{-B(2; 2; 2) \cdot 3}{2} = 1$

Riemann Zeta function - $\zeta(x) = \frac{1}{\Gamma(x)} \cdot \int_{0}^{1} \frac{(-\ln(t))^{x-1}}{1 - t} \, \mathrm{d}t$

[6; 5.59]



[1.2; 0]

Checks:

$Precision = 10^{-15}$

$\zeta(2) - \frac{\pi^2}{6} = \zeta(2) - \frac{3.14^2}{6} = 2.22\times10^{-16}$

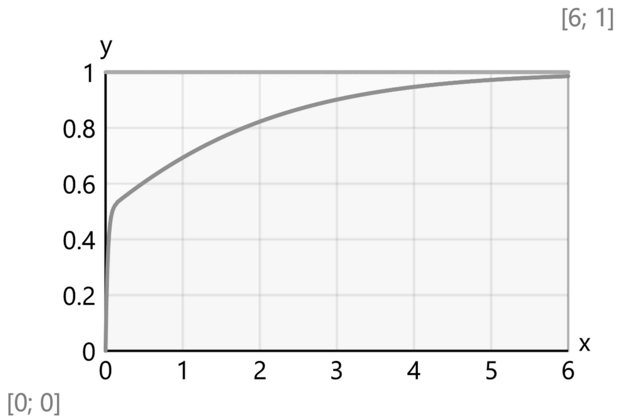
$\zeta_3 = 1.2$ - Apéry constant

$\zeta(3) - \zeta_3 = \zeta(3) - 1.2 = 4\times10^{-15}$

$\zeta(4) - \dfrac{\pi^4}{90} = \zeta(4) - \dfrac{3.14^4}{90} = 0$

$Precision = 10^{-4} = 0.0001$

Dirichlet Eta function - $\eta(x) = \dfrac{1}{\Gamma(x)} \cdot \displaystyle\int_0^1 \dfrac{(-\ln(t))^{x-1}}{1+t}\,dt$



Checks:

$\eta_{0.5} = 0.605$

$\eta(0.5) - \eta_{0.5} = \eta(0.5) - 0.605 = 4.64\times10^{-8}$

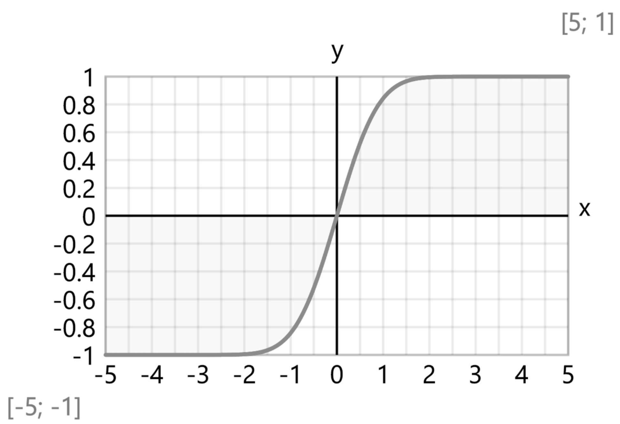
$Precision = 10^{-15}$

$\eta(1) - \ln(2) = 7.77\times10^{-16}$

$\eta(2) - \dfrac{\pi^2}{12} = \eta(2) - \dfrac{3.14^2}{12} = -2.22\times10^{-16}$

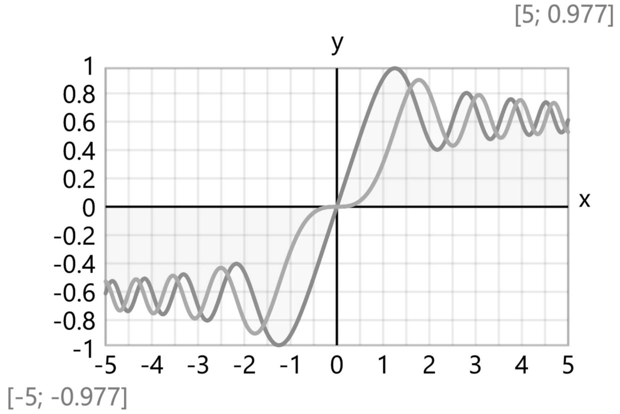
$\eta(4) - \dfrac{7\cdot\pi^4}{720} = \eta(4) - \dfrac{7\cdot3.14^4}{720} = -1.11\times10^{-16}$

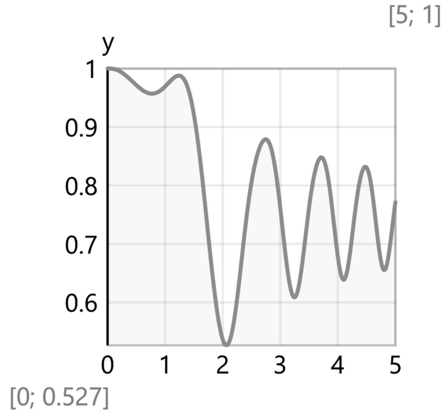
Error function - $erf(x) = \dfrac{2\cdot\mathbf{sign}(x)}{\sqrt{\pi}} \cdot \displaystyle\int_0^{|x|} e^{-(t^2)}\,dt$, $erfc(x) = 1 - erf(x)$



Integral functions

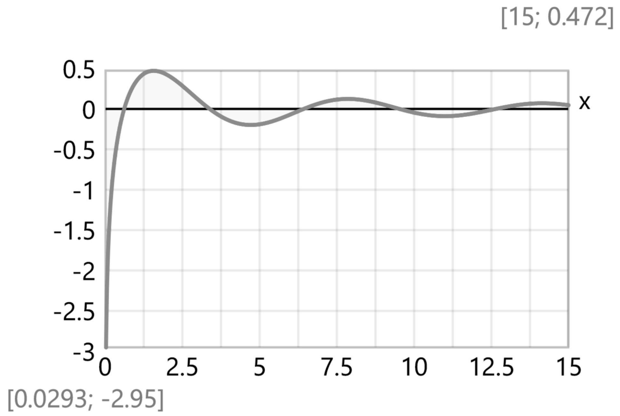
Fresnel integrals: $C(x) = \int_0^x \cos(t^2)\,\mathrm{d}t$, $S(x) = \int_0^x \sin(t^2)\,\mathrm{d}t$

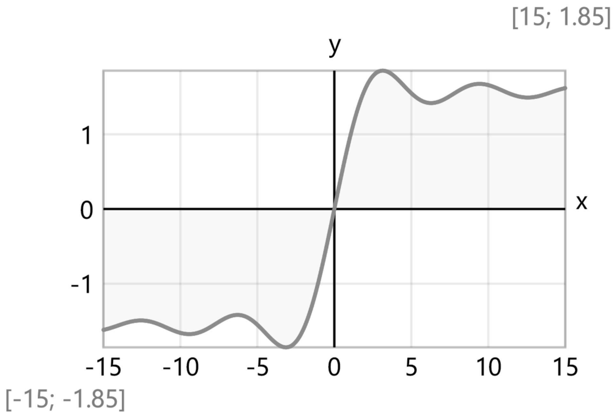




Sine and cosine integrals:

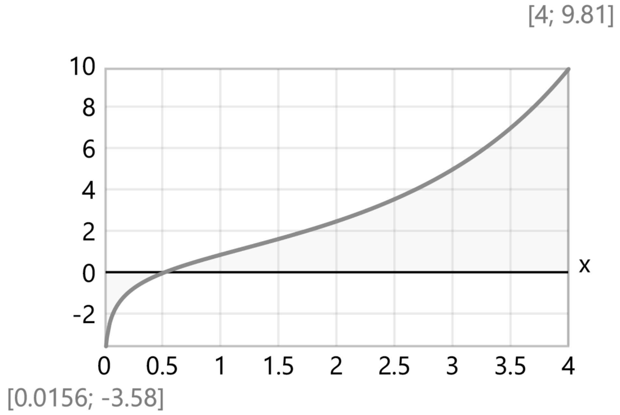
$$Ci(x) = \begin{cases} \text{if } x \equiv 0\text{: } -1/0 \\ \text{else: } \gamma + \ln(x) + \int_0^x \dfrac{\cos(t) - 1}{t}\,\mathrm{d}t \end{cases}, \quad Si(x) = \begin{cases} \text{if } x \equiv 0\text{: } 0 \\ \text{else: } \int_0^x \dfrac{\sin(t)}{t}\,\mathrm{d}t \end{cases}$$

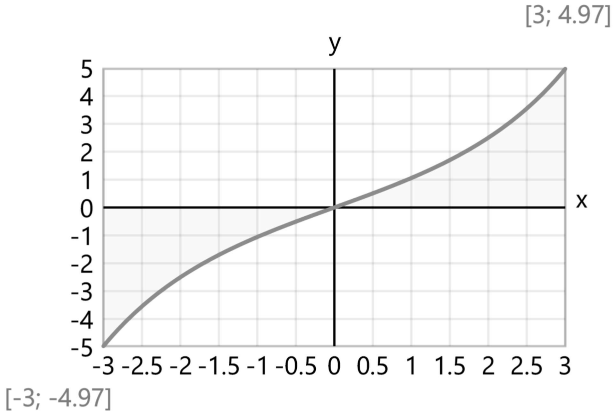




Hyperbolic sine and cosine integrals:

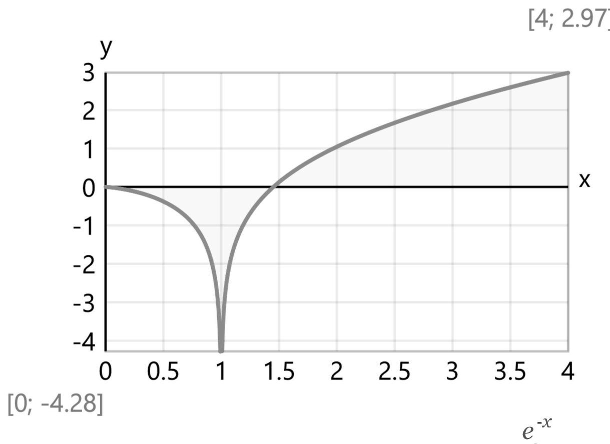
$$Chi(x) = \begin{cases} \text{if } x \equiv 0\text{: } -1/0 \\ \text{else: } \gamma + \ln(x) + \int_0^x \dfrac{\cosh(t) - 1}{t}\,\mathrm{d}t \end{cases}, \quad Shi(x) = \begin{cases} \text{if } x \equiv 0\text{: } 0 \\ \text{else: } \int_0^x \dfrac{\sinh(t)}{t}\,\mathrm{d}t \end{cases}$$



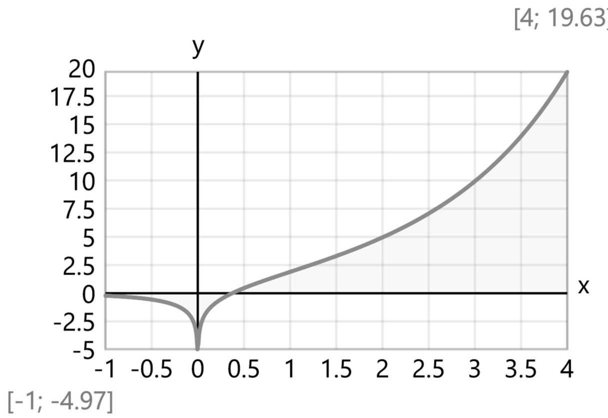


$li_2 = 1.05$

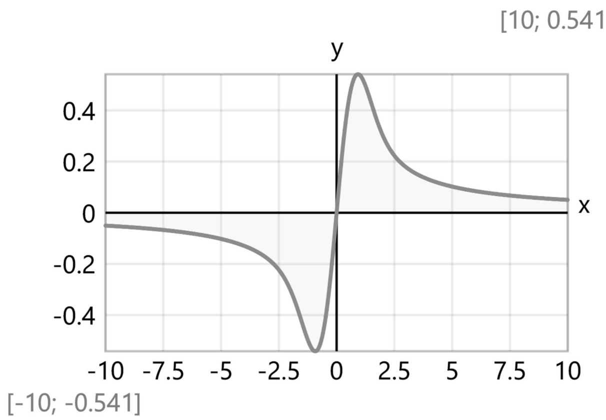
Logarithmic Integral - $Li(x) = \int_2^x \frac{1}{\ln(t)}\,\mathrm{d}t$, $li(x) = \begin{cases} \text{if } x < 1: \int_0^x \frac{1}{\ln(t)}\,\mathrm{d}t \\ \text{else: } Li(x) + li_2 \end{cases}$



Exponential Integral - $E_1(x) = -\int_0^{e^{-x}} \frac{1}{\ln(t)}\,\mathrm{d}t$, $Ei(x) = \begin{cases} \text{if } x < 0: -E_1(-x) \\ \text{else if } x > 0: li_2 + Li(e^x) \end{cases}$



Dawson's Integral - $F_\mathrm{D}(x) = e^{-(x^2)} \cdot \int_0^x e^{t^2}\,\mathrm{d}t$

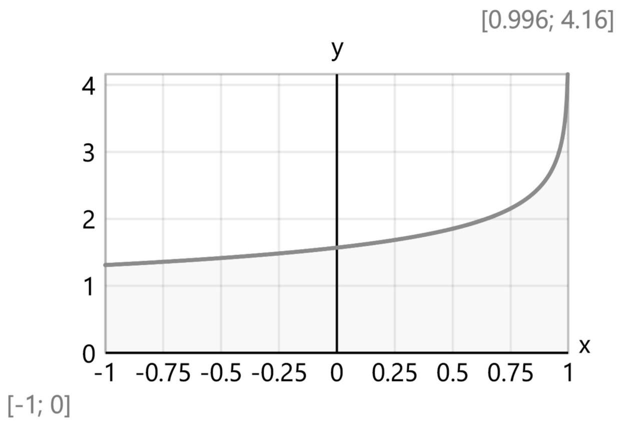


Elliptic integrals

Incomplete elliptic integral of the first kind -

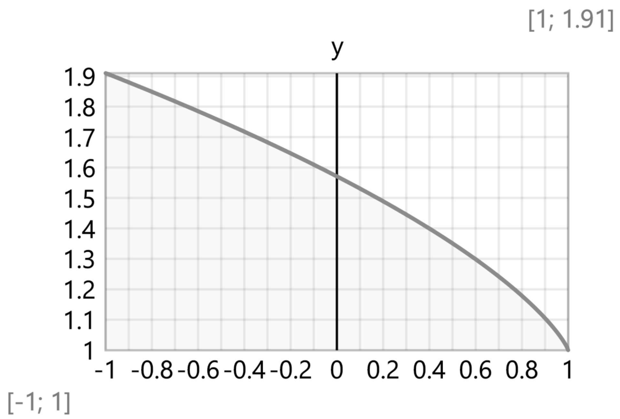
$$F(\varphi; m) = \int_0^{\varphi} \frac{1}{\sqrt{\max(1 - m \cdot \sin(t)^2; \varepsilon)}}\,\mathrm{d}t$$

Complete elliptic integral of the first kind - $K(m) = F\left(\frac{\pi}{2}; m\right)$



Incomplete elliptic integral of the second kind - $E\_(\varphi; m) = \int_0^{\varphi} \sqrt{1 - m \cdot \mathbf{sin}(t)^2}\, \mathrm{d}t$

Complete elliptic integral of the second kind - $E(m) = E\_\left(\frac{\pi}{2}; m\right)$

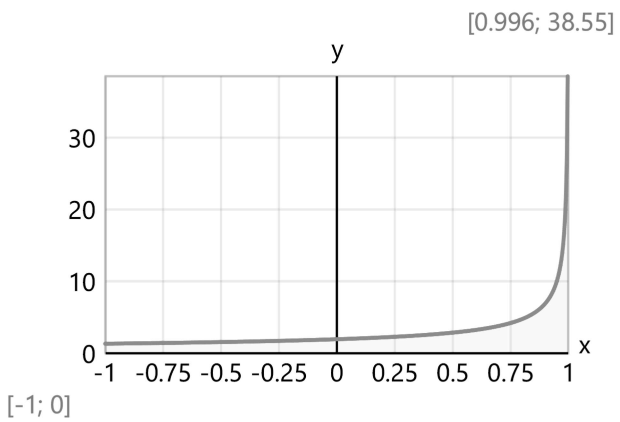


Incomplete elliptic integral of the third kind -

$$\Pi\_(n; \varphi; m) = \int_0^{\varphi} \frac{1}{\mathbf{max}(1 - n \cdot \mathbf{sin}(t)^2; \varepsilon) \cdot \sqrt{\mathbf{max}(1 - m \cdot \mathbf{sin}(t)^2; \varepsilon)}}\, \mathrm{d}t$$

Complete elliptic integral of the third kind $\Pi(n; m) = \Pi\_\left(n; \frac{\pi}{2}; m\right)$

$\Pi(0.4; 0.6) = 2.59$



Jacobi elliptic functions

Jacobi elliptic amplitude

Gudermannian function - $gd(x) = \mathbf{asin}(\mathbf{tanh}(x))$ = am(u; 1)

Approximate value for small *u*, *m*, *m'*

$$am\_(u; m) = gd(u) + \frac{1 - m}{4} \cdot (\mathbf{sinh}(u) \cdot \mathbf{cosh}(u) - u) \cdot \mathbf{sech}(u)$$

To avoid numerical instabilities, the function for larger values of u is reduced to the interval [0; K(m)] where the elliptic integral is evaluated within [0; π/2]. This is performed by using the following quasi-periodical relationships:

am(*u* + 2K(*m*), *m*) = **am**(*u*, *m*) + π, for *u* ≥ 2K(*m*)
am(*u*, *m*) = π - **am**(2K(*m*) - *u*, *m*), for *u* < 2K(*m*)

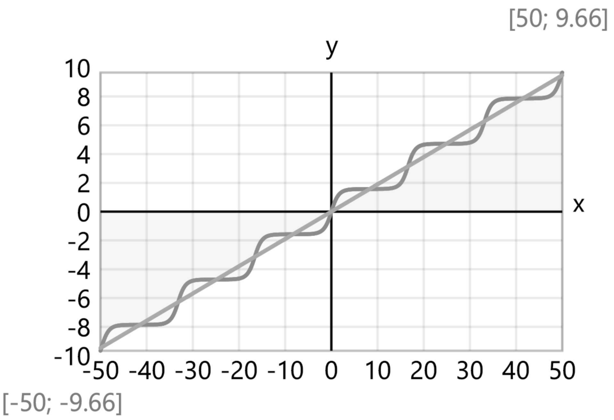
Function for evaluation of Jacobi elliptic amplitude:

$am(u; m) =$
if $m \equiv 0$: u
else if $m \equiv 1$: $\mathbf{asin}(\mathbf{tanh}(u))$
else:

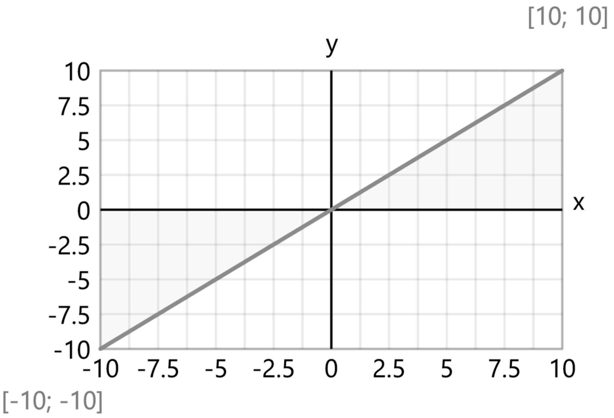
$$K_{m2} = K(m) \cdot 2$$
$$s = \mathbf{sign}(u)$$
$$u = |u|$$
$$n = \mathbf{floor}\left(\frac{u}{K_{m2}}\right)$$
$$u = u - n \cdot K_{m2}$$
$$y = \frac{u \cdot \pi}{K_{m2}}$$
$$am = \begin{cases} \text{if } u > \dfrac{K_{m2}}{2}: \pi - \$\text{Root}\{F(\varphi; m) = K_{m2} - u; \varphi \in [0; \dfrac{\pi}{2}]\} \\ \text{else: } \$\text{Root}\{F(\varphi; m) = u; \varphi \in [\mathbf{max}(0; y - 1); \mathbf{min}\left(\dfrac{\pi}{2}; y + 1\right)]\} \end{cases}$$
$$s \cdot (am + n \cdot \pi)$$

Plot for $m = 1$, $1 - m = 1 - 1 = 10^{-6}$

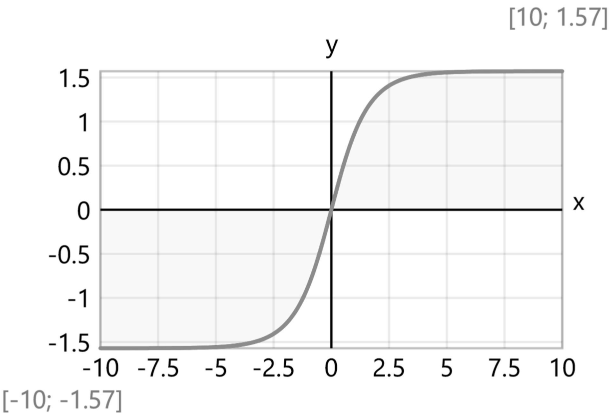
$$Precision = \frac{10^{-15}}{\sqrt[3]{1 - m}^2} = \frac{10^{-15}}{\sqrt[3]{1 - 1}^2} = 10^{-11}$$



Check: **am**(*x*; 0) = x



Check: **am**(*x*; 1) = **gd**(*x*)



Jacobi elliptic functions

$sn(u; m) = \mathbf{sin}(am(u; m))$

$cn(u; m) = \mathbf{cos}(am(u; m))$

$dn(u; m) = \begin{cases} sn = \mathbf{sin}(am(u; m)) \\ \sqrt{1 - m \cdot sn \cdot sn} \end{cases}$

$cs(u; m) = \begin{cases} \varphi = am(u; m) \\ \dfrac{\mathbf{cos}(\varphi)}{\mathbf{sin}(\varphi)} \end{cases}$

$cd(u; m) = \begin{cases} \varphi = am(u; m) \\ sn = \mathbf{sin}(\varphi) \\ cn = \mathbf{cos}(\varphi) \\ dn = \sqrt{1 - m \cdot sn \cdot sn} \\ cn/dn \end{cases}$

$dc(u; m) = \begin{cases} \varphi = am(u; m) \\ sn = \mathbf{sin}(\varphi) \\ cn = \mathbf{cos}(\varphi) \\ dn = \sqrt{1 - m \cdot sn \cdot sn} \\ dn/cn \end{cases}$

$$sc(u; m) = \begin{cases} \varphi = am(u; m) \\ \dfrac{\sin(\varphi)}{\cos(\varphi)} \end{cases}$$

$$sd(u; m) = \begin{cases} \varphi = am(u; m) \\ sn = \sin(\varphi) \\ dn = \sqrt{1 - m \cdot sn \cdot sn} \\ sn/dn \end{cases}$$

$$ds(u; m) = \begin{cases} \varphi = am(u; m) \\ sn = \sin(\varphi) \\ dn = \sqrt{1 - m \cdot sn \cdot sn} \\ dn/sn \end{cases}$$

$Precision = 10^{-15}$

Checks:

$am(1; 0.5) = 0.932$, $dn(1; 0.5) = 0.823$

$s = sn(1; 0.5) = 0.803$, $c = cn(1; 0.5) = 0.596$

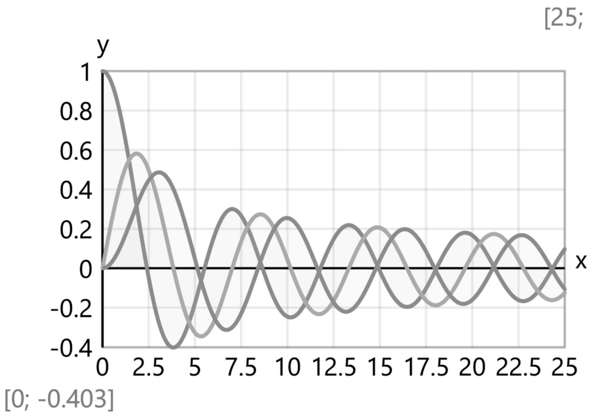
$c^2 + s^2 = 0.596^2 + 0.803^2 = 1$

Reciprocal Jacobi elliptic functions

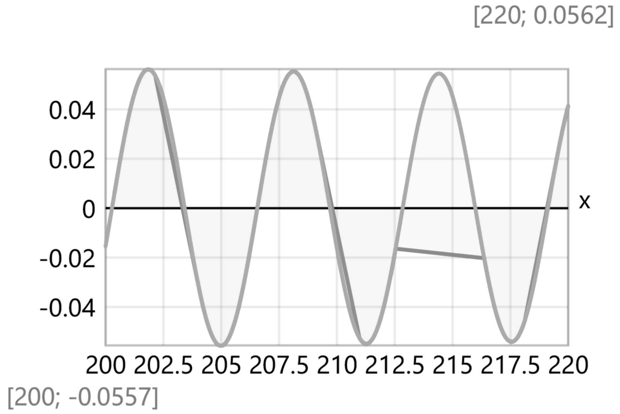
$ns(u; m) = \dfrac{1}{sn(u; m)}$, $nc(u; m) = \dfrac{1}{cn(u; m)}$, $nd(u; m) = \dfrac{1}{dn(u; m)}$

Bessel functions

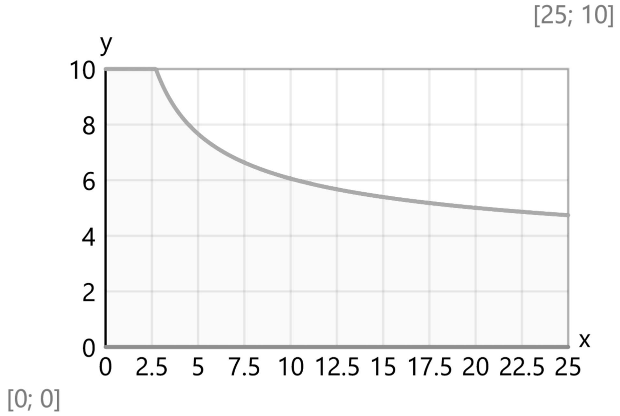
Bessel functions of the first kind - $J(n; x) = \dfrac{1}{\pi} \cdot \int_0^{\pi} \cos(n \cdot \theta - x \cdot \sin(\theta))\, d\theta$



Asymptotic expansion (use for x > 150) - $J_a(n; x) = \sqrt{\dfrac{2}{\pi \cdot x}} \cdot \cos\left(x - \dfrac{n \cdot \pi}{2} - \dfrac{\pi}{4}\right)$



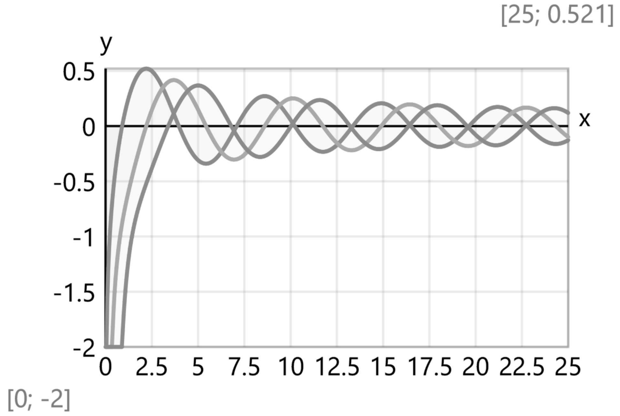
Dynamic limit for numerical stability - $inf(x) = \dfrac{20}{1 + \mathbf{ln}(\mathbf{max}(x; e))}$

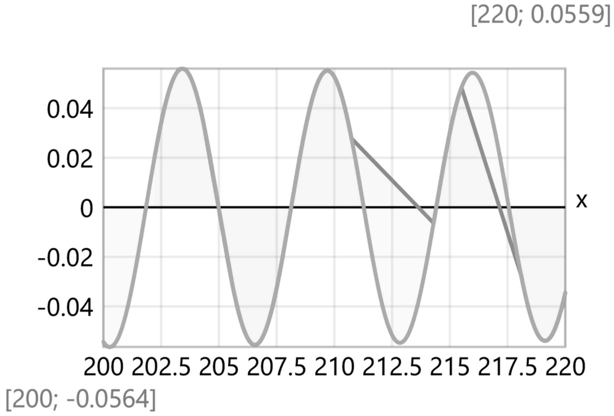


Bessel functions of the second kind

$$Y(n; x) = \frac{1}{\pi} \cdot \int_0^{\pi} \mathbf{sin}(x \cdot \mathbf{sin}(\theta) - n \cdot \theta)\, \mathrm{d}\theta - \frac{1}{\pi} \cdot \int_0^{inf(x)} (\mathbf{exp}(n \cdot t) + (-1)^{n} \cdot \mathbf{exp}(-n \cdot t)) \cdot \mathbf{exp}(-x \cdot \mathbf{sinh}(t))\, \mathrm{d}t$$

Asymptotic expansion (use for x > 150) - $Y_{\mathrm{a}}(n; x) = \sqrt{\dfrac{2}{\pi \cdot x}} \cdot \mathbf{sin}\left(x - \dfrac{n \cdot \pi}{2} - \dfrac{\pi}{4}\right)$

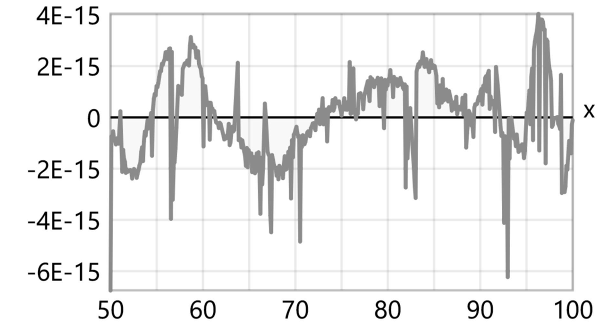




Recurrence test: $n = 1$, $x = 100$

$$Y(n - 1; x) + Y(n + 1; x) - \frac{2 \cdot n}{x} \cdot Y(n; x) = Y(1 - 1; 100) + Y(1 + 1; 100) - \frac{2 \cdot 1}{100} \cdot Y(1; 100) = -1.11\times10^{-16}$$

[100; 4.04E-15]



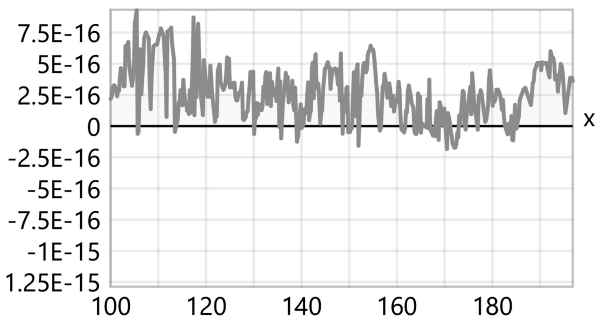
[50; -6.76E-15]

Wronskian test: W(x) = $J_1(x) \cdot Y_0(x) - J_1(x) \cdot Y_0(x) = 2 / (\pi \cdot x)$

$$Wr(x) = J(1; x) \cdot Y(0; x) - J(0; x) \cdot Y(1; x)$$

$$Wr(x) - \frac{2}{\pi \cdot x} = Wr(100) - \frac{2}{3.14 \cdot 100} = 2.18 \times 10^{-16}$$

[196.88; 9.3E-16]



[100; -1.29E-15]

Modified Bessel functions of the first kind

Modified Bessel functions of the second kind

Airy functions

Lambert W function

Helper function - $ln_2(x) = \mathbf{ln}(\mathbf{ln}(x))$

Approximate value - $W_a(x) = \mathbf{ln}(x) - ln_2(x)$

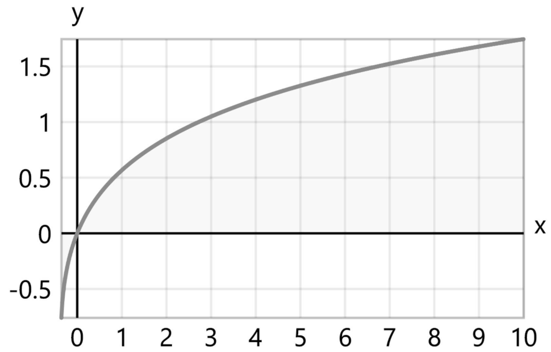
Secondary value - $W_b(x) = \dfrac{ln_2(x)}{\mathbf{ln}(x)}$

Lower bound - $W_{btm}(x) = W_a(x) + 0.5 \cdot W_b(x)$

Upper bound - $W_{top}(x) = W_a(x) + \dfrac{e}{e-1} \cdot W_b(x)$

The function - $W(x) = \begin{cases} \text{if } x < e\text{: } \$\text{Root}\{\xi \cdot \mathbf{exp}(\xi) = x;\ \xi \in [-1;\ 1]\} \\ \text{else: } \$\text{Root}\{\xi \cdot \mathbf{exp}(\xi) = x;\ \xi \in [W_{btm}(x);\ W_{top}(x)]\} \end{cases}$

[10; 1.75]



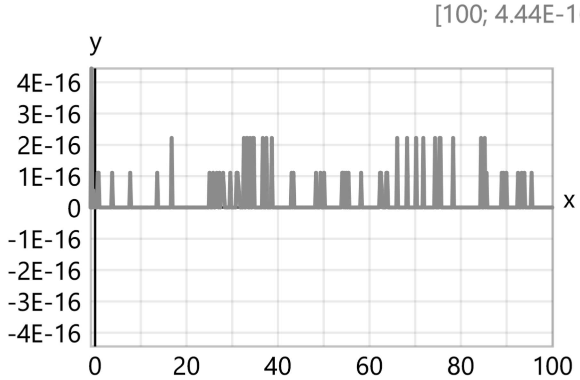
[-0.355; -0.76]

Omega constant - $\Omega = 0.567$

Check: $W(1) - \Omega = W(1) - 0.567 = 0$, $W(e) - 1 = W(2.72) - 1 = 0$

Relative error plot

[100; 4.44E-16]



[-1; -4.44E-16]